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Log-Convex Polynomial Approximation in Log. of Twice Differentiable Functions.

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Abstract

In this paper, we introduce a new concept of constrained approximation, which we call Log-convex polynomial approximation in log. We obtained a direct theorem of log-convex polynomial approximation in log for twice differentiable functions on [-1, 1], in term of the ordinary moduli of continuity.

Keywords: Shape preserving approximation, convex and log-convex functions, Splines, moduli of continuity and smoothness.

1.Introduction and Main results.

$C^k[a,b]$ is the space of k^{th} time continuously differentiable functions f on $[a,b]$, equipped with the uniform norm $\|f\|_{[a,b]} := Max_{x\in[a,b]}|f(x)|$. When dealing with the generic interval $I := [-1,1]$, we omit the special reference to the interval, namely, we write $\|f\| := \|f\|_I$. Also, let π_n be the space of all algebraic polynomials of degree not exceeding n, where n and k are natural numbers.

We state some definitions in this topic, first of all, a function $f \in C(I)$ is said to be a convex function [1], if for each $x, y \in I$ and $t \in [0,1]$, the function f satisfies:

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

Also, a function $f : I \to (0,\infty)$ is said to be a logarithmically convex (or log-convex function) [1] and [2], if for each $x, y \in I$ and $t \in [0,1]$ one has the following inequality:

$$f(tx + (1-t)y) \le [f(x)]^t [f(y)]^{(1-t)},$$

or equivalently(see [3]),a function $f \in C(I)$ is called log-convex, if f is positive and $\log f$ is convex on I. Furthermore, we have: (for example see [3]):

Theorem A. If $f \in C^2(I)$, then it is log-convex if and only if $f(x) > 0$ and

$$f(x)f''(x) \ge [f'(x)]^2, \forall x \in I.$$

The r^{th} ordinary moduli of smoothness is defined by (see [4]):

1.1 $$\omega_r(f,t) := \underset{0<h<t}{Sup}\left\| \Delta_h^r(f,\cdot) \right\|, \ t > 0$$

where

$$\Delta_u^r(f,x) := \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f\left(x - \frac{ru}{2} + iu\right), & if \left| x \mp \frac{ru}{2} \right| < 1, \\ 0 & otherwise, \end{cases}$$

is the r^{th} symmetric forward difference of f. Note that, if $r = 1$, then it called the ordinary moduli of continuity.

Some time one want to approximate a continuous function $f \in C(I)$ by a polynomials such that

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those polynomials have the same shape property as f has, and this is the topic which so called constrained (or shape preserving or form preserving (see[5]) approximation. Questions of this nature have been investigated extensively in recent year (see [6] for more detail). For monotone and convex approximation, there are so many good results (for example see [6-8]). However, there are some few results about 3-covex approximation by spline functions (for example see [9]). While more notable result on convex approximation for our work is the following theorem due to the work of Yu [10]:

Theorem B. Let f be a convex function in $C(I)$. Then for each $n \geq 2$ there exists a convex polynomial $P_n \in \pi_n$ such that

1.2 $\quad \|f - P_n\| \leq C\omega_2\left(f, n^{-1}\right),$

where C is an absolute constant independent of f , n,

Remark 1. ω_2 can not replace by ω_4, while keeping the constant C independent of n and f , see Shvedov [11].

It is quite natural to one ask what about the log-convex polynomial approximation (i.e., approximating a log-convex function $f \in C(I)$ by a log-convex polynomial)? This is our main interest in this paper; especially we will introduce a new type of this concept, which we will call log-convex polynomial approximation in log, and by it we mean, approximation of a log-convex function by a log-convex polynomial such that the maximum error in them logs at each point are controlled, we do this for these log-convex functions which contained in $C^2(I)$, exactly our main results in this area are the following theorem and corollary:

Theorem 1. Let f be a log-convex function in $C^2(I)$. Then for each $n \geq 2$, there exists a log-convex polynomial $\mathrm{P}_n \in \pi_n$, such that

1.3 $\quad \|\log f - \log \mathrm{P}_n\| \leq C n^{-2}\omega\left((\log f)'', \frac{1}{n}\right)$, where C is an absolute constant independent of f and n.

Corollary 2. Let S be a log-convex Spline in $C^k(I)$, $k \geq 2$ of any degree with Chebyshev partition of the interval I as its knots. Then for each $n \geq 2$ there exists a log-convex polynomial $\mathrm{P}_n \in \pi_n$, such that

1.4 $\quad \|\log S - \log \mathrm{P}_n\| < C\omega_3\left(\log S, n^{-1}\right),$

where C is an absolute constant independent of f and n.

Throughout this paper C denote absolute constants which is independent of f and n. Also may be different on different occurrence, even on the same line. At the same time the absolute constants C_1 and C_2 are fixed throughout this paper. Furthermore, we say that two constants A and B are equivalent, and we denote by $A \sim B$, if there are absolute constants c and c' such that $cA \leq B \leq c'B$.

2. Auxiliaries Construction and Lemmas.

Given a twice-differentiable log-convex function f on I, that is f is a positive function and $\log f$ is convex function in $C^2(I)$. Then, we will divide the interval I into two types of intervals I_i and J_i in the following way, we begin with:

$$O := \left\{ x \in I \; ; f(x) f''(x) > [f'(x)]^2 + 6C_1 \omega\left((\log f)'' , n^{-1} \right) \right\},$$

where C_1 is an integer greater than or equal to three, and it was prescribed in[12, lemma 4], after replacing a convex function in it by a log-convex function.

Now, since O is an open interval in the restriction topology on I . So by using [13, Theorem 8.11] we can represent O as the union of a countable collection of disjoint intervals. Then we need the following sets, which are introduced by Hu, Leviatan and Yu [12],

$$O^* := \bigcup_{\beta_j - \alpha_j \geq C_2 n^{-1}} (\alpha_j, \beta_j) := \bigcup_{j=1}^{n_1} I'_j .$$

where $(\alpha_j, \beta_j) := I'_j$'s are the intervals which made O and are of relatively large size, $n_1 \leq \dfrac{2n}{C_2}$ and C_2 is a positive such that $C_2 \geq 4(dn + 6C_1)$, d is given in Lemma 2.4 below. With $\beta_0 := -1$ and $\alpha_{n_1+1} := 1$, let

2.1 $$O_1 := \bigcup_{\alpha_{j+1} - \beta_j \geq 12 C_1 n^{-1}} [\beta_j, \alpha_{j+1}] = \bigcup J_j ,$$

where $[\beta_j, \alpha_{j+1}] := J_j$'s are selected from the completing of the intervals I'_j's and

2.2 $$O_2 := I - O_1 =: \bigcup_{i=1}^{n_2} I_i ,$$

where $I_i := (a_i, b_i)$ are the original intervals which represented O, after we only renamed the indices so that the intervals $J_i := [b_i, a_{i+1}]$ in O_1 and $n_2 \leq n_1$. Any I_i is either one of the intervals I'_j or it contains in several of them with its endpoints belonging to the original set of endpoints of the I'_j , and not necessary to be an original pair. Thus $b_i - a_i \geq C_2 n^{-1}$ and

2.3 $$(\log f)''(a_i) = (\log f)''(b_i) =: 6C_1 \omega\left((\log f)'' , n^{-1} \right) \cdot$$

Now, let $I_i =: \left(\bigcup I'_{i,j} \right) \cup \left(\bigcup I''_{i,j} \right)$, where $I'_{i,j}$ are contained in O^* and $I''_{i,j}$ are the complementing once. Also if $x \in J_i$, then $x \in I - O$ or $x \in O$, that is $x \in I - O$ or x is in one of the original intervals (α_i, β_i) that represented O, with $\beta_j - \alpha_j < C_2 n^{-1}$. Then, we have

2.4 $$(\log f)''(x) \leq (6C_1 + C2) \omega\left((\log f)'' , n^{-1} \right), \quad x \in O_1$$

Hence, we can decompose $(\log f)''$ in the following way:
We define, for each $i = 1,2,\dots,n_2$,

2.5 $$f_i(x) := \begin{cases} (\log f)''(x) & x \in I_i , \\ (\log f)''(x) R_i(x) & x \in [a_i - 6C_1 n^{-1}, a_i], \\ (\log f)''(x) \tilde{R}_i(x) & x \in [b_i, b_i + 6C_1 n^{-1}], \\ 0 & o.w, \end{cases}$$

where

2.6 $$R_i(x) := \begin{cases} 1 & x = a_i - 6C_1 n^{-1}, \\ \dfrac{x - a_i}{6C_1 n^{-1}} + 1 & x \in (a_i - 6C_1 n^{-1}, a_i), \\ 0 & x = a_i , \end{cases}$$

and

2.7 $$\tilde{R}_i(x) := \begin{cases} 1 & x = b_i , \\ 1 - \dfrac{x - b_i}{6C_1 n^{-1}} & x \in (b_i, b_i + 6C_1 n^{-1}), \\ 0 & x = b_i + 6C_1 n^{-1}. \end{cases}$$

Also, we define:

2.8 $$f_0(x) := (\log f)''(x) - \sum_{i=1}^{n_2} f_i(x).$$

From the fact that $f(x) f''(x) \geq [f'(x)]^2$, we have, for each $i = 1,2,\dots,n_2$, $f_i \geq 0$ and $f_i \in C(I)$. Furtherermore, their supports have disjoint interiors. At the same time, we can prove the following lemma:

Lemma 2.1 We have

$$0 \le f_0(x) \le (C_2 + 6C_1)\omega\left((\log f)'', n^{-1}\right), \ \forall x \in I.$$

Proof: let $x \in I$, then $x \in O_1$ or $x \in O_2$.

If $x \in O_2$. Then, in view of (2.1), $x \in (a_k, b_k)$ for some $1 \le k \le n_2$. Thus $f_k(x) = (\log f)''(x)$ and $f_i = 0$ for each $i \ne k$. Therefore, by (2.8) we have $f_0(x) = 0$. Hence, the lemma is proved in this case. For the other case (i.e., if $x \in O_1$), then in virtue of (2.5) through (2.7), there is at most one $1 \le i \le n_2$ such that $f_i(x) \ne 0$ and all other f_i's are identically zero. Hence, by (2.8) for such x, we have

$$f_0(x) = (\log f)''(x) - f_i(x).$$

Nevertheless, since $0 \le R_i(x), \tilde{R}_i(x) \le 1$, then in view of (2.5) we notice that $f_0(x) \ge 0$. Now, since f_i is non-negative, then

$$f_0(x) \le (\log f)''(x) \le (C_2 + 6C_1)\omega\left((\log f)'', n^{-1}\right),$$

where we used (2.4) in the last inequality. ◊

The following lemma is well known:

Lemma 2.2. [14]

For a function $g \in C(I)$, we have

$$\omega_r(g, nt) \le n^r \omega_r(g, t).$$

Then, we can prove the following lemma:

Lemma 2.3. We have,

$\omega\left(f_i, n^{-1}\right) \le 3\omega\left((\log f)'', n^{-1}\right)$, for each $i = 1, 2, \ldots, n_2$.

Proof: let $x_1, x_2 \in I$ such that $|x_1 - x_2| \le n^{-1}$. Then we will use the subintervals, which we have in (2.5), and we begin from right to left:

(i) If $b_i + 6C_1 n^{-1} > x_1, x_2$. Then

$$f_i(x_1) = 0 = f_i(x_2), .$$

Hence, there is nothing to prove.

(ii) If $b_i < x_1 < b_i + 6C_1 n^{-1} \le x_2$. Then, $1 - \frac{x_1 - b_i}{6C_1 n^{-1}} \le \frac{x_2 - x_1}{6C_1 n^{-1}}$, so that,

$$\tilde{R}_i(x_1) \le \frac{x_2 - x_1}{6C_1 n^{-1}} \le (6C_1)^{-1}.$$

and we observe that, for any $x \in [b_i, b_i + 6C_1 n^{-1}]$

$$\begin{aligned}(\log f)''(x) &\le (\log f)''(b_i) + \left|(\log f)''(x) - (\log f)''(b_i)\right| \\ &\le 6C_1\omega\left((\log f)'', n^{-1}\right) + \omega\left((\log f)'', 6C_1 n^{-1}\right) \\ &\le 12C_1\omega\left((\log f)'', n^{-1}\right),\end{aligned} \tag{2.9}$$

where, we used (1.1) with $r = 1$, (2.3) and Lemma 2.2.

Hence, in virtue of (2.5) and (2.9) we have $f_i(x_2) = 0$ and

$$\begin{aligned}|f_i(x_1) - f_i(x_2)| &= f_i(x_1) = (\log f)''(x_1)\tilde{R}_i(x_1) \\ &\le 2C_1\omega\left((\log f)'', n^{-1}\right).\end{aligned}$$

(iii) If $x_1, x_2 \in [b_i, b_i + 6C_1 n^{-1}]$, then

$$\begin{aligned}|f_i(x_1) - f_i(x_2)| &= \left|(\log f)''(x_1)\tilde{R}_i(x_1) - (\log f)''(x_2)\tilde{R}_i(x_2)\right| \\ &\le \left|(\log f)''(x_1) - (\log f)''(x_2)\right|\tilde{R}_i(x_1) \\ &\quad + (\log f)''(x_2)\left|\tilde{R}_i(x_1) - \tilde{R}_i(x_2)\right| \\ &\le \omega\left((\log f)'', n^{-1}\right) + 12C_1\omega\left((\log f)'', n^{-1}\right)\left|\frac{x_1 - x_2}{6C_1 n^{-1}}\right| \\ &\le 3\omega\left((\log f)'', n^{-1}\right),\end{aligned}$$

where, we used (1.1) with $r = 1$, (2.9)and the fact that $|x_1 - x_2| \le n^{-1}$.

(iv) If $x_1 \in [a_i, b_i]$ and $x_2 \in [b_i, b_i + 6C_1 n^{-1}]$.

Then

$$|f_i(x_1)-f_i(x_2)|=\left|(\log f)''(x_1)-(\log f)''(x_2)\tilde{R}_i(x_2)\right|$$
$$\le\left|(\log f)''(x_1)-(\log f)''(x_2)\right|\tilde{R}_i(b_i)$$
$$+(\log f)''(x_2)\left|\tilde{R}_i(x_1)-\tilde{R}_i(b_i)\right|$$
$$\le\omega\left((\log f)'',n^{-1}\right)+\left(6C_1\omega\left((\log f)'',n^{-1}\right)+\omega\left((\log f)'',n^{-1}\right)\right)\frac{1}{6C_1}$$
$$\le 2\omega\left((\log f)'',n^{-1}\right)+\frac{1}{6C_1}\omega\left((\log f)'',n^{-1}\right)$$
$$\le 3\omega\left((\log f)'',n^{-1}\right)$$

where, we used (1.1) with $r=1$, (2.3), (2.5), (2.7), the fact that $C_1\ge 3$ and $x_2-b_i\le x_1-x_2\le n^{-1}$.

(v) If $x_1,x_2\in[a_i,b_i]$, we have

$$|f_i(x_1)-f_i(x_2)|=\left|(\log f)''(x_1)-(\log f)''(x_2)\right|$$
$$\le \sup_{\substack{|x-y|<n^{-1}\\ x,y\in I}}\left|(\log f)''(x)-(\log f)''(y)\right|$$
$$=\omega\left((\log f)'',n^{-1}\right),$$

where we used (2.5) and (1.1) with $r=1$.

By using (2.6) in place of (2.7), we can get the same inequalities in similar way, in the case that x's are on the left side of the interval $[a_i,b_i]$. This completes the proof. ◊

Now, for each $i=0,1,\ldots,n_2$ we define:

$$2.10 \qquad F_i(x):=\int_{-1}^{x}\int_{-1}^{u} f_i(v)\,dv\,du\,,$$

and it is clear that, $F_i\in C^2(I)$ and it is convex, for each $i=0,1,\ldots,n_2$. Thus, we have a decomposition of $\log f$ in the form of

$$2.11 \qquad \log f(x)=\log f(-1)+\frac{(x+1)f'(-1)}{f(-1)}+\sum_{i=0}^{n_2}F_i(x).$$

Let $d_n(x,J)=1+nd(x,J)$, where J is an interval and $d(x.J)$ is the distance from x to J. Now, since f is log-convex function, then $\log f$ is convex. Thus, as an immediate consequence of [12, Lemma 4], we obtain the following lemma:

Lemma 2.4. There exists a number $d\sim n^{-1}$ and a polynomial $q(x)$ of degree not exceeding n such that in $[-2,2]$,

$$2.12 \qquad q(x)\ge 3C_1\omega\left((\log f)'',n^{-1}\right)d_n(x,\bar{J})^{-4},\ |x|>d\,,$$

and

$$2.13 \qquad q(x)\ge -3C_1\omega\left((\log f)'',n^{-1}\right),\ |x|\le d\,.$$

Also

$$2.14 \qquad \left|\int_{-2}^{x}\int_{-2}^{u}q(v)\,dv\,du\right|<Cn^{-2}\omega\left((\log f)'',n^{-1}\right)d_n(x,\bar{J})^{-2},|x|\le 2,$$

where $\bar{J}:=[-d-12C_1n^{-1},d+12C_1n^{-1}]$, C is only de-pend on C_1.

3.Proof of Theorem 1 and Corollary 2. First, proof of Theorem 1.

Since $F_0\in C^2(I)$ and it is convex in I, so in virtue of theorem B and Lemma 2.1, there exist a convex polynomial $P_n\in\pi_n$ such that

$$\|F_0-P_n\|\le Cn^{-2}\|F_0''\|\le Cn^{-2}\omega\left((\log f)'',n^{-1}\right),$$

Let us denote this polynomial by $P_{0,n}$.

Now, by Lemma 2.3 and Lemma 2.4 a polynomial $P_{in}\in\pi_n$ exists, such that

$$3.1 \qquad |F_i(x)-P_{in}(x)|\le 3C_1n^{-2}\omega\left((\log f)'',n^{-1}\right)d_n(x,\hat{J}_i)^{-2},$$

and

3.2 $\left|F_i''(x)-P_{in}''(x)\right| \le 3C_1\omega\left(\left(\log f\right)'', n^{-1}\right) d_n\left(x,\hat{J}_i\right)^{-4}$
,

where $\hat{J} := [a_i - 12C_1n^{-1}, b_i + 12C_1n^{-1}]$.

Now, since $I_i = \left(\bigcup I'_{i,j}\right) \cup \left(\bigcup I''_{i,j}\right)$, so for any $x \in I_{i,j}$ (as we mentioned above) $\left(\log f\right)''(x) \ge 6C_1\omega\left(\left(\log f\right)'', n^{-1}\right)$, so that, by (3.2) we obtain $P_{in}''(x) \ge 3C_1\omega\left(\left(\log f\right)'', n^{-1}\right)$,

We to modify P_{in} in order to get a convex polynomial and we do this modification only to these polynomials P_{in} which may have a negative second derivative in $I'_{i,j}$. To this, we set:

$$Q(x) := \int_{-2}^{x}\int_{-2}^{u} q(v)\,dv\,du, \quad |x| \le 2,$$

where q is the polynomial in Lemma 2.4 and we choose the constant C_2 so that $C_2 \ge 4(dn + 6C_1)$ (as we motioned in section 1). With $I'_{i,j} = (a_{i,j}, b_{i,j})$, $j = 1,2,...m_i$, and $i = 1,2,...,n_2$. We define:

3.3 $q_i(x) := Q(x - (a_i + d)) + \sum_{j=1}^{m_i} Q(x + d - b_{i,j}), \; x \in I,$

where $b_{i,m_i} = b_i$ and if $a_1 = -1$, b_{n_2} $b_{n_2} = 1$ we omit the first term and the last term in the sum, respectively.

Now, we notice that by our assumption for $a_{i,j}, b_{i,j}$, we have

3.4 $$\begin{aligned} b_{i,j} - a_{i,j} &\ge C_2n^{-1} \ge 2d, \\ a_{i,j+1} - b_{i,j} &\le 12C_1n^{-1}. \end{aligned}$$

Hence, for each $x \in \left(\hat{J}_i - \left(\bigcup I'_{i,j}\right)\right)$, we have $x \notin I'_{i,j}$ for all i, j. So that, by first part of (3.4) we have $|x| \ge d$ and then by (2.12) we obtain

3.5 $q_i''(x) \ge 3C_1\omega\left(\left(\log f\right)'', n^{-1}\right)$,

For $x \in I'_{i,j}$, then by second part of (3.4) and (2.13) we obtain

3.6 $q_i''(x) \ge -3C_1\omega\left(\left(\log f\right)'', n^{-1}\right)$.

Finally, for $x \notin \hat{J}_i$, we have $x < a_i - 12C_1n^{-1}$ or $x > b_i + 12C_1n^{-1}$.

Note that, for $x < a_i - 12C_1n^{-1}$, we have $d_n(x, J_{i,0}) = d_n(x, \hat{J}_i)$, where $J_{i0} := [a_i - 12C_1n^{-1}, 2d + a_i + 12C_1n^{-1}]$ and for the case $x > b_i + 12C_1n^{-1}$, we have $d_n(x, J_{i,m_i}) = d_n(x, \hat{J}_i)$, where $J_{i,m_i} := [-2d + b_i - 12C_1n^{-1}, b_i + 12C_in^{-1}]$. Hence in both cases $|x| > 2d$, then (2.12) implies that

3.7 $q_i''(x) \ge 3C_1\omega\left(\left(\log f\right)'', n^{-1}\right) d\left(x, \hat{J}_i\right)^{-4}$,

for $x \in \hat{J}_i$,

Then, in virtue of (2.13) we have

$$\left|Q(x + d - b_{i,j})\right| \le Cn^{-2}\omega\left(\left(\log f\right)'', n^{-1}\right) d_n\left(x, J_{i,m_i}\right), \; x \in I$$

where $J_{i,j} := [-2d + b_{i,j} - 12C_1n^{-1}, b_{i,j} + 12C_1n^{-1}]$.

Analogously

$$\left|Q(x - (a_i + d))\right| \le Cn^{-2}\omega\left(\left(\log f\right)'', n^{-1}\right) d_n\left(x, J_{i,0}\right), \; x \in I$$

In view of the first part of (3.4) and the definition of C_2, for $0 < j \le m_i$, $0 < i \le n_2$,

$$\begin{aligned} d_n\left(J_{i,j}, J_{i,j-1}\right) &:= b_{i,j} - b_{i,j-1} - 2d - 24C_1n^{-1} \\ &\ge b_{i,j} - a_{i,j} - 2d - 24C_1n^{-1} \\ &\ge C_2n^{-1} - 2d - 24C_1n^{-1} \\ &\ge 2d \ge 0, \end{aligned}$$

and

$$d\left(J_{i,0}, J_{i,1}\right) = b_{i,1} - a_i - 4d - 24C_1 n^{-1}$$
$$\geq C_2 n^{-1} - 4d - 24C_1 n^{-1} \geq 0.$$

This mean that the distance between two consecutive $J_{i,j}$ and $J_{i,j+1}$ make their interiors disjoint for all i and j. Therefore, although J_{i,m_i} may intersect $J_{i+1,0}$, we have

3.8 $$\left|\sum q_i(x)\right| \leq Cn^{-1}\omega\left(\left(\log f\right)'', n^{-1}\right)\sum_{i,j} d_n\left(x, J_{i,j}\right)^{-2}$$

$$\leq Cn^{-2}\omega\left(\left(\log f\right)'', n^{-1}\right).$$

Set

$$\mathrm{P}_n(x) := \log f(-1) + (x+1)(\log f)'(-1) + P_{0n} + \sum_{i=1}^{n_2} P_{in} - q_i.$$

Note that, in view of our construction of the polynomials P_{in} and q_i, the polynomial P_n is very close to the logarithmic function, so we can say;

$$\log \mathrm{P}_n(x) := \log f(-1) + (x+1)(\log f)'(-1) + P_{0n} + \sum_{i=0}^{n_2} P_{in} - q_i$$

Now, in virtue of (3.2), (3.5) through (3.7) and the convexity of $\log f$, it follows that $P_{in} + q_i$ are convex for each $i = 1,2,\ldots,n_2$, this with the fact that P_{0n} is convex.

Hence,

$$\left\|\log f - \log \mathrm{P}_n\right\| \leq \left\|F_0 - P_{0n}\right\| + \left\|\sum_{i=1}^{n_2} F_i - P_{in}\right\| + \left\|\sum_{i=1}^{n_2} q_i\right\|$$
$$\leq Cn^{-2}\omega\left(\left(\log f\right)'', n^{-1}\right). \Diamond$$

Second, proof of Corollary 2:

Combining [15, Theorem 1] with our Theorem 1, we obtain the assertion of our corollary 2. ◊

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نزیکردنەوەی فرەرادەداری لۆگاریتمە-قوپاو لە لۆگدا، بۆ نەخشە جیاکارییە دوو هێندەکان.

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پوختە

لەم توێژینەوەدا هەستاوین بە دروست کردنی جۆرێکی نوێ لە نزیکردنەوەی مەرجدار (شێوە پاراستن) کە ناوی دەبەین بە نزیکردنەوەی فرەرادەداری لۆگاریتمە-قوپاو لە لۆگدا، وە تیوێری ڕاستەوخۆ بۆ پێوانەکردنی فرەڕادەداری لۆگاریتمە-قوپاو لە لۆگدا بۆ نەخشە دووهێندە جیاکارییە لۆگاریتمە-قوپاوەکانی سەر [-1، 1]، بە بەکارهێنانی (پێوانەی بەردەوامی ئاسایی)مان بەدەست هێناوە.

تقريب متعددات الحدود المحدبة-اللوغاريتمية في اللوغاريتم، للدوال القابلة لتفاضل مرتين.

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